Directed polymer and percolation models

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Classic example: random walk.

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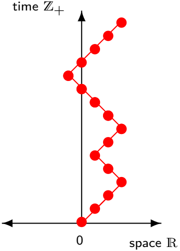

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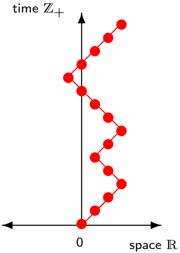

What does S_n look like at large times n?

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Suggests: typical fluctuations of S_n of order $n^{1/2}$.

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Central limit theorem

$$\lim_{n\to\infty} P\left\{ \frac{\bar{S}_n}{\sigma\sqrt{n}} \le s \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-x^2/2}\, dx$$

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Summary: all RWs with finite second moment share

- limiting velocity $S_n/n \to E(X_1)$
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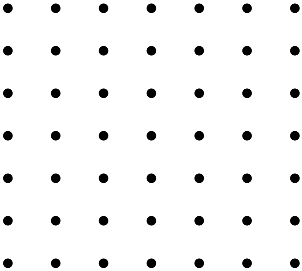
Last two are an example of **universality.**

Directed percolation



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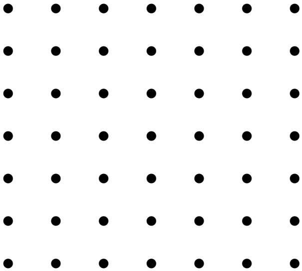

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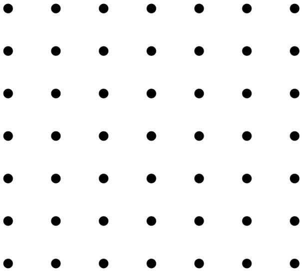

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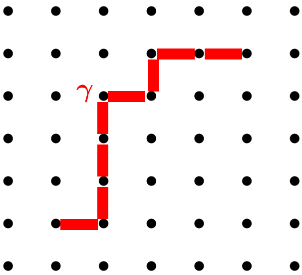

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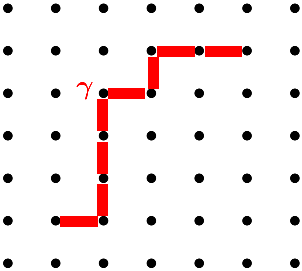

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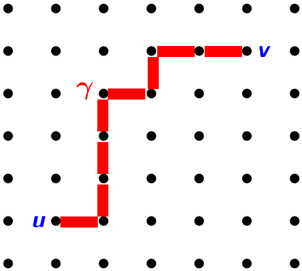

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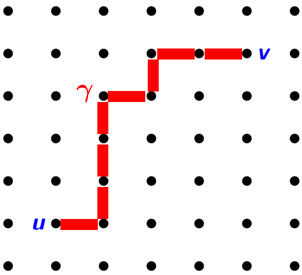

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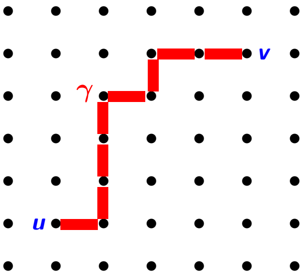

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Called **last-passage percolation** (LPP) because slowest path wins.

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Theorem. Assume $\mathbb{E}|\omega_x|^{2+\varepsilon} < \infty$. $\exists$ deterministic $g : \mathbb{R}_+^2 \to \mathbb{R}$ s.t.

$$\lim_{n\to\infty} n^{-1} G_{0,\lfloor n\xi \rfloor} = g(\xi) \qquad \text{a.s.} \qquad \forall\, \xi \in \mathbb{R}_+^2$$

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Ergodic theory gives only an asymptotic representation for the limit:

$$g(\xi) \;=\; \sup_n \, n^{-1}\, \mathbb{E}[G_{0,\,n\xi}\,] \qquad \text{for } \xi \in \mathbb{N}^2.$$

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Any other characterization for g ?

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LPP is the zero-temperature polymer:

$$\lim_{\beta \to \infty} \beta^{-1} \log Z_{\boldsymbol{u},\boldsymbol{v}} = \max_{\gamma : \boldsymbol{u} \to \boldsymbol{v}} W(\gamma) = G_{\boldsymbol{u},\boldsymbol{v}}.$$

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Large deviation theory treats asymptotics of this type.

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(S. Varadhan Abel Prize 2007)

Warm-up: periodic environment

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where $\rho(A) \; = \;$ **Perron-Frobenius eigenvalue** of A.

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P2ℓ limiting free energy $\quad g_{\text{pl}} = \log \rho(A) \quad$ for

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Infimum over **gradients**, achieved at right e-vector $\varphi = e^f$.

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Centered cocycles $=$ L^1 closure of gradients

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Directed polymer: formula for p2ℓ free energy

Recall the p2ℓ limiting free energy of the directed polymer:

$$g_{\text{pl}}^{\beta} = \lim_{n\to\infty} n^{-1}\,\beta^{-1}\log \sum_{\gamma_0=0,\,|\gamma|=n} 2^{-n}\exp\Big\{\beta\sum_{x\in\gamma}\omega_x\Big\}$$

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Theorem (Rassoul-Agha, S, Yılmaz, 2013)

$$g_{\rm pl}^{\beta} = \inf_{F}\ \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega}\ \beta^{-1}\log\sum_{i=1,2}\tfrac{1}{2}\,e^{\beta\omega_0+\beta F(\omega,0,e_i)}$$

where infimum over centered stationary L^1 cocycles F. Minimizer exists.

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$$g^{\beta}_{\text{pl}} = \inf_{F}\ \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega}\ \beta^{-1}\log\sum_{i=1,2}\tfrac{1}{2}\,e^{\beta\omega_0+\beta F(\omega,0,e_i)}$$

where infimum over centered stationary L^1 cocycles F. Minimizer exists.

Proof Quenched large deviation principle for the empirical measure $n^{-1}\sum_{k=0}^{n-1}\delta_{(T_{X_k}\omega,\,X_{k+1}-X_k)}$ under a fixed ω, with X_k = RW.

Ergodic properties of cocycles.

Return to point-to-line last-passage percolation

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Theorem (Georgiou, Rassoul-Agha, S, 2013+)

$$g_{\mathrm{pl}}^{\infty} = \inf_{\boldsymbol{F}} \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega} \max_{i\in\{1,2\}} \{\omega_0 + \boldsymbol{F}(\omega, 0, e_i)\}$$

where the infimum is over centered stationary L^1 cocycles $\boldsymbol{F}$. Minimizer exists.

Variational formulas for FPP and LPP

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Ours is probabilistic as explained:

- variational formula for polymers via large deviation theory (though we borrow technical ideas from homogenization work of Kosygina and Varadhan)
- zero-temperature limit.

So we have these variational formulas...

Directed polymer:

$$g_{\mathrm{pl}}^{\beta} = \lim_{n\to\infty} n^{-1} \log \sum_{(x_k)_{k=0}^{n}} 2^{-n} \exp\Big\{ \sum_{k=0}^{n-1} \omega_{x_k} \Big\}$$

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with infima over centered stationary L^1 cocycles F.

Are there interesting models where we can understand what goes on in these formulas ?

Exactly solvable models

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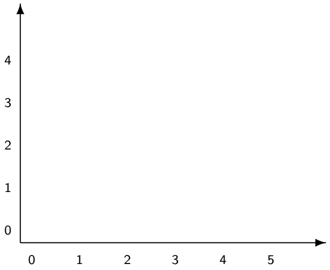
- **Tractable stationary version** of the $Z_{\boldsymbol{u},\boldsymbol{v}}$ process.

Stationary log-gamma polymer



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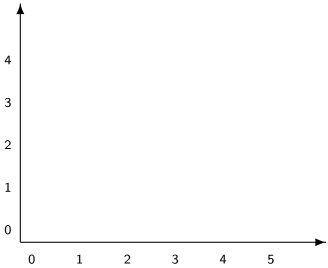

Two parameters $0 < \alpha < \rho$.

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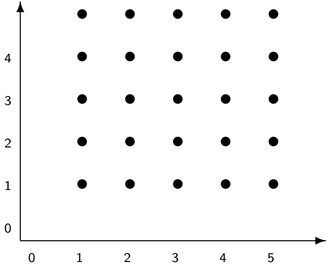

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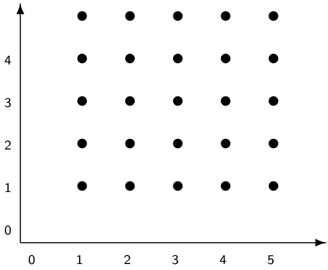

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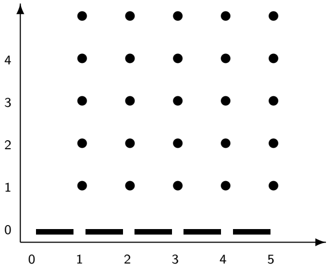

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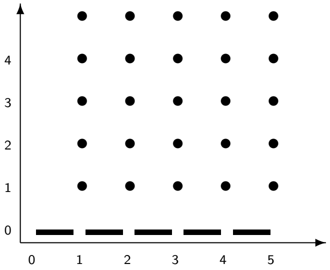

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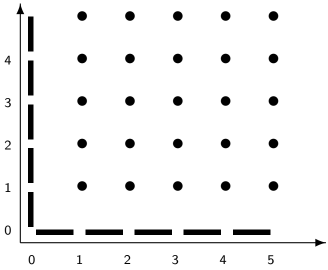

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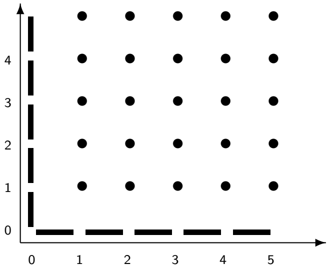

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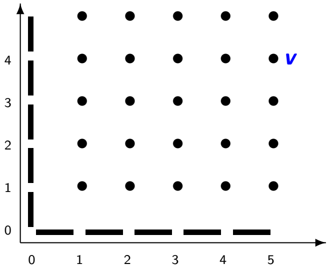

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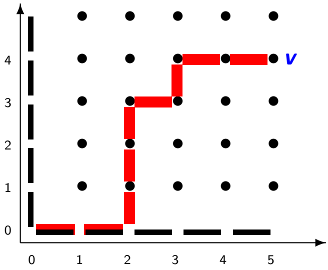

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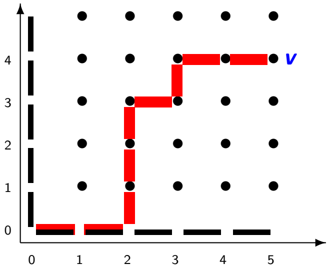

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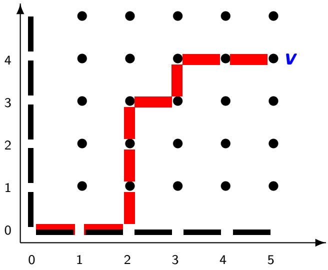

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$$Z^{\alpha}_{\mathbf{0},v} = \sum_{x(\cdot):\, \mathbf{0}\to v} \left(\prod_{k=1}^{T} \tau^{-1}_{x(k-1),\, x(k)} \right) \cdot \left(\prod_{k=T+1}^{|v|_1} w^{-1}_{x(k)} \right)$$

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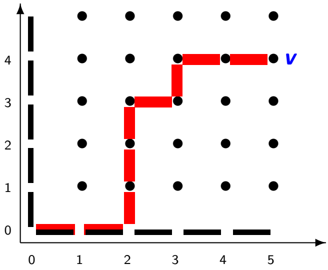

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where T is the exit time of the path from the boundary.

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Tractable steady states useful for calculations and proofs.

Next some consequences.

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where direction ξ picks parameter $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\xi) \in (0,\rho)$ via

$$\xi \;\cdot\; \bigl(\Psi_1(\boldsymbol{\alpha})\,,\, -\Psi_1(\rho - \boldsymbol{\alpha})\bigr) \;=\; 0.$$

$\Psi_0 = \Gamma'/\Gamma, \quad \Psi_1 = \Psi_0'.$

Busemann functions solve variational formulas for log-gamma polymer

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For other directions ξ cocycles F^{ξ} solve related variational problems, including the point-to-point cases.

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Presently such results only proved for exactly solvable models.

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$\star$ **Possible End** $\star$

LPP in periodic environment

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This leads to $g_{\text{pl}}^{\infty} = \lambda$ and e-value equation is

$$g_{\text{pl}}^{\infty} \;=\; \max_{i=1,2}\,[\,\omega_0 \,+\, \sigma(T_{e_i}\omega) \,-\, \sigma(\omega)\,]$$

so variational formula for g_{pl}^{∞} minimized by $\sigma(T_{e_1}\omega) - \sigma(\omega)$.

Variational formulas in terms of measures

Recall the limits:

$$g_{\mathrm{pl}}^{\beta}(h) \;=\; \lim_{n\to\infty}\; n^{-1}\,\log \sum_{(x_k)_{k=0}^{n}} 2^{-n}\,\exp\Big\{\sum_{k=0}^{n-1}\omega_{x_k} \;+\; h\cdot x_n\Big\}$$

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Pick random step e_i, jump from $\eta=(\omega,z)$ to $S_{e_i}\eta = (T_z\eta, e_i)$.

Entropy relative to the Markov kernel

For measures $\boldsymbol{\mu}$ and transition kernels $\boldsymbol{q}$ on $\Omega \times \{e_1, e_2\}$, familiar relative entropy

$$H(\boldsymbol{\mu} \times \boldsymbol{q} \,|\, \boldsymbol{\mu} \times p) \;=\; \int_{\Omega \times \{e_1, e_2\}} \sum_{i=1,2} \boldsymbol{q}(\eta, S_{e_i}\eta) \log \frac{\boldsymbol{q}(\eta, S_{e_i}\eta)}{p(\eta, S_{e_i}\eta)} \, \boldsymbol{\mu}(d\eta).$$

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Let $\boldsymbol{\mu}_0 = \Omega$-marginal of $\boldsymbol{\mu}$. Define

$$H_{\mathbb{P}}(\boldsymbol{\mu}) = \begin{cases} \inf \left\{ H(\boldsymbol{\mu} \times \boldsymbol{q} \,|\, \boldsymbol{\mu} \times p) : \boldsymbol{\mu}\boldsymbol{q} = \boldsymbol{\mu} \right\} & \text{if } \boldsymbol{\mu}_0 \ll \mathbb{P} \\ \infty & \text{otherwise.} \end{cases}$$

Infimum over Markov kernels $\boldsymbol{q}$ that fix $\boldsymbol{\mu}$.

Variational formulas

$\mathcal{M}_s(\Omega \times \{e_1, e_2\}) \;=\;$ space of invariant measures $\boldsymbol{\mu}$:

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$$g_{\text{pl}}^{\beta}(h) \;=\; \sup\Big\{ E^{\boldsymbol{\mu}}[\,\omega_0 + h \cdot Z\,] - \beta^{-1} H_{\mathbb{P}}(\boldsymbol{\mu}) : \boldsymbol{\mu} \in \mathcal{M}_s(\Omega \times \{e_1, e_2\}),\, \boldsymbol{\mu}_0 \ll \mathbb{P},\, E^{\boldsymbol{\mu}}[\omega_0^-] < \infty \Big\}$$

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Trouble: condition $\boldsymbol{\mu}_0 \ll \mathbb{P}$ removes compactness.