On the entropy of sums in continuous and discrete settings

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Based on joint works with M.Fradelizi, I.Kontoyiannis and M.Rapaport

Outline

Entropy

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$$\begin{aligned}
p(X_1, \ldots, X_n) &= p^{\#\{i:X_i=H\}}(1-p)^{\#\{i:X_i=T\}} \\
&= 2^{\#\{i:X_i=H\}\log p + \#\{i:X_i=T\}\log(1-p)} \\
&= 2^{n\left(\frac{\#\{i:X_i=H\}}{n}\log p + \frac{\#\{i:X_i=T\}}{n}\log(1-p)\right)} \\
&\approx 2^{n(p\log p + (1-p)\log(1-p))} \\
&= 2^{-n(-p\log p - (1-p)\log(1-p))}
\end{aligned}$$

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$$\Rightarrow p(X_1, \ldots, X_n) \approx 2^{-nH(X)} \text{ with high probability}$$

Entropy

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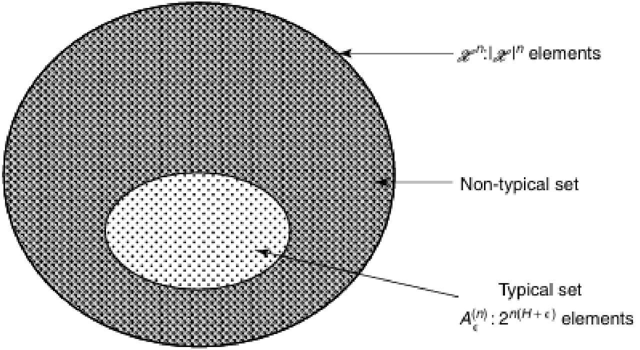

FIGURE 3.1.

Entropy

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Shannon entropy:

$$H(X) = -\sum_{x \in S} p(x) \log p(x)$$

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$$0 \leq H(X) \leq \log |\mathrm{supp}(X)| = H(U), \quad U \sim \mathrm{Uni}(\mathrm{supp}(X))$$

Entropy Power Inequality

Differential entropy: of $X \sim f$ on $\mathbb{R}^d$

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- $h(AX) = h(X) + \log |\det(A)|$ Scaling
- $h(X) \leq \frac{d}{2} \log \left(2\pi e \det(\mathrm{Cov}(X))^{\frac{1}{d}} \right) = h(Z),\ Z \sim \mathcal{N}(\mathbb{E}X, \mathrm{Var}(X))$
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$$|A_n| \approx e^{nh(X)}$$ Volume of typical set

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- Goes back to Shannon (1948)
- Fundamental consequences in communications
- Geometry: Brunn-Minkowski inequality

$$\mathrm{Vol}(A+B)^{\frac{1}{d}} \geq \mathrm{Vol}(A)^{\frac{1}{d}} + \mathrm{Vol}(B)^{\frac{1}{d}}, \quad A, B \subset \mathbb{R}^d$$

Entropy Power Inequality (EPI)

Say $X_1 \in B_d(0, r)$, $X_2 \in B_d(0, R)$ with d large
With high probability $X_1 + X_2 \in B_d(0, \sqrt{r^2 + R^2})$ (by orthogonality)

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In other words, while $B_d(0, r) + B_d(0, R) = B_d(0, r + R)$

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EPI:

$$|A \overset{99\%}{+} B|^{1/d} \geq \sqrt{|A|^{2/d} + |B|^{2/d}}$$

Source: https://mathoverflow.net/questions/167951/entropy-proof-of-brunn-minkowski-inequality

Entropy Power Inequality and the CLT

Entropy Power Inequality (EPI) for i.i.d. on $\mathbb{R}$

$$h(X_1 + X_2) \geq h(X_1) + \frac{1}{2}\log 2$$

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or by scaling

$$h\Big(\frac{X_1 + X_2}{\sqrt{2}}\Big) \geq h(X_1)$$

First step of monotonicity in the entropic CLT [Barron '86]:

$$h\Big(\frac{\sum_{i=1}^n X_i}{\sqrt{n}}\Big) \to h(Z) = \frac{1}{2}\log\left(2\pi e \mathrm{Var}(X_1)\right) \Rightarrow \mathrm{CLT}$$

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[Artstein, Ball, Barthe, Naor '04]:

$$h\Big(\frac{\sum_{i=1}^{n+1} X_i}{\sqrt{n+1}}\Big) \geq h\Big(\frac{\sum_{i=1}^n X_i}{\sqrt{n}}\Big)$$

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BUT $H(X_1 + X_2) \geq H(X_1) + \frac{1}{2}\log 2$ fails in general

Discrete Entropy Power Inequality

However,

Theorem (Tao, 2010)

Let X be a RV on a finite subset of a torsion-free *group $(\mathbb{Z}, \mathbb{Z}^d, \mathbb{R}$, etc.) Then*

$$H(X_1 + X_2) \geq H(X_1) + \frac{1}{2}\log 2 - o(1)$$

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- Additive combinatorics: $A \subset \mathbb{Z}$, $A + A := \{a_1 + a_2 : a_1, a_2 \in A\}$
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- CLT!!!
- "Discrete Gaussians" have the smallest entropy doubling

Discrete Entropy Power Inequality

Conjecture [Tao, '10]

For any $n \geq 1$

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- "Approximate" monotonicity

Discrete Entropy Power Inequality

Definition

A random variable X with PMF p on $\mathbb{Z}$ is called *log-concave* if

$$p(k)^2 \geq p(k-1)p(k+1) \quad \text{for every } k \in \mathbb{Z}$$

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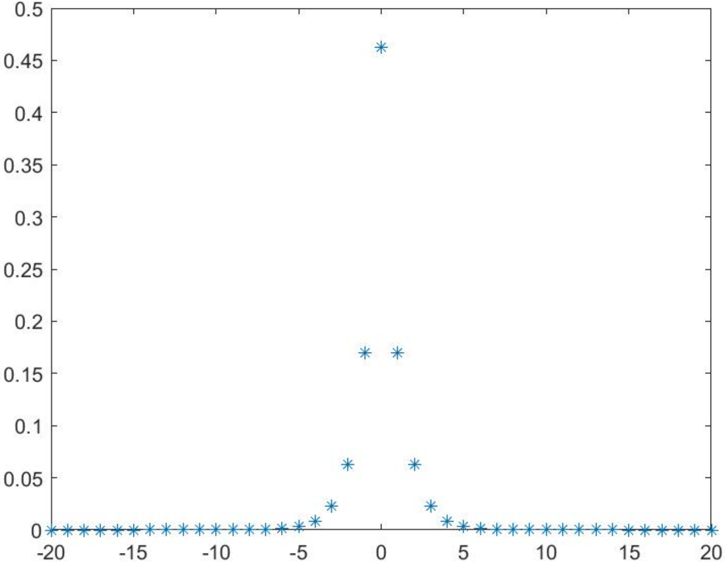
But includes many important distributions, e.g. Bernoulli, Binomial, Poisson, Geometric, Uniform,...

Discrete Entropy Power Inequality



Discrete Entropy Power Inequality

Theorem (G. 2023)

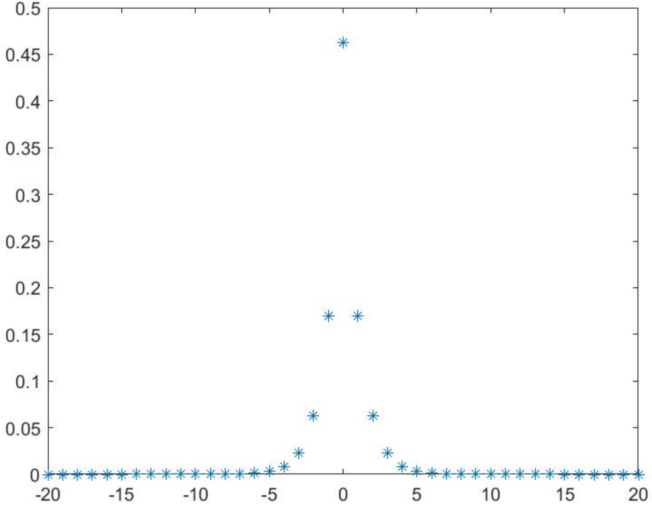
Let $n \geq 1$ *and* $\epsilon > 0$*. Suppose* $X_1, \ldots, X_n$ *are i.i.d. log-concave random variables on the integers. Then*

$$H(X_1 + \cdots + X_{n+1}) \geq H(X_1 + \cdots + X_n) + \frac{1}{2} \log\left(\frac{n+1}{n}\right) - \epsilon,$$

provided that $H(X_1) \geq \log \frac{2}{\epsilon} + \log\log \frac{2}{\epsilon} + n + 27$

Discrete Entropy Power Inequality

$$p(k) \propto \frac{1}{\lambda} e^{-\lambda|k|}$$



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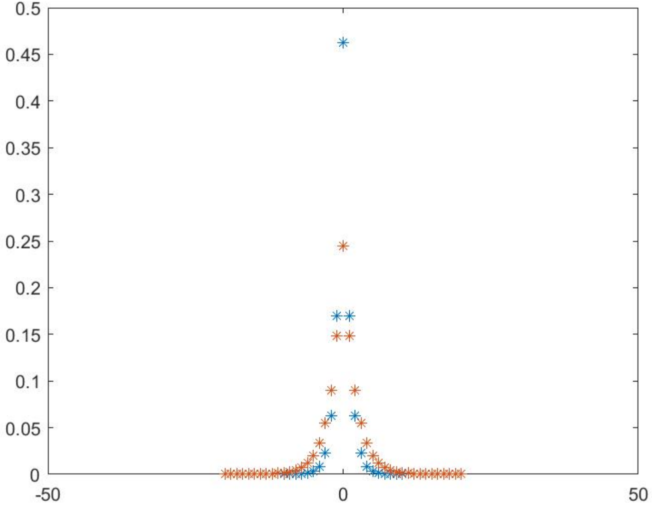

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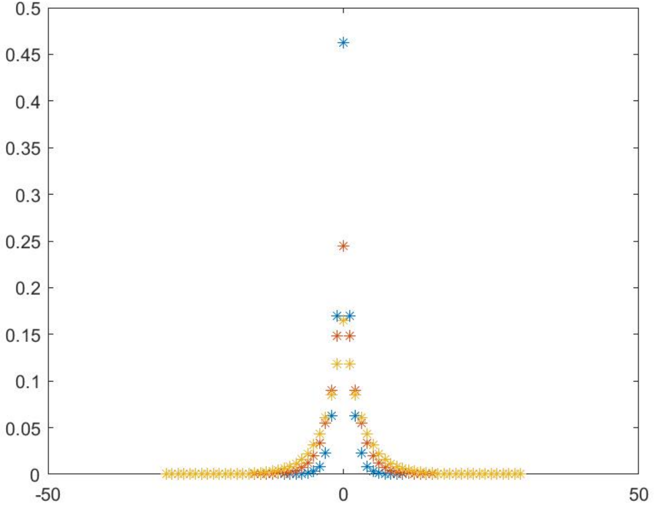

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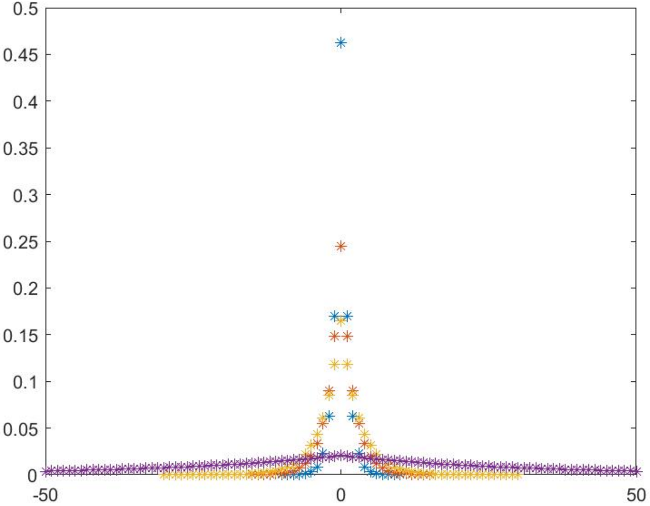

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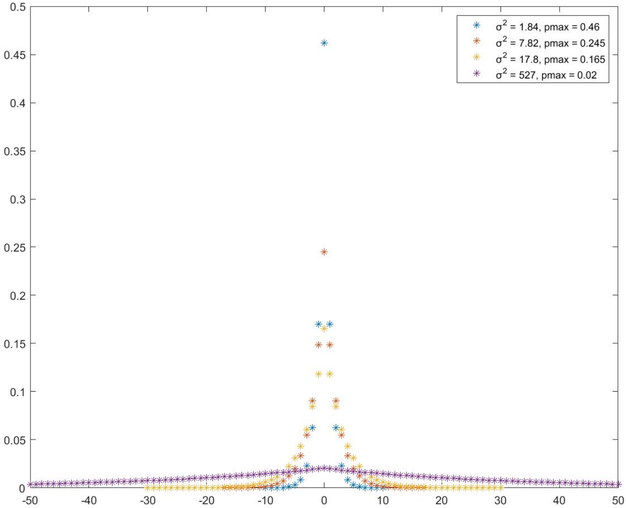

Discrete Entropy Power Inequality

Theorem (Fradelizi, G., Rapaport (2024))

For any i.i.d. log-concave random vectors $X_1, \ldots, X_{n+1}$ *on* $\mathbb{Z}^d$

$$H(X_1 + \cdots + X_{n+1}) \geq H(X_1 + \cdots + X_n) + \frac{1}{2}\log\left(\frac{n+1}{n}\right) - o(1)$$

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If, in addition, X_i *have *almost isotropic extension then*

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- $o(1) = O\left(H(X_1) e^{-\frac{1}{d} H(X_1)}\right)$
- Holds under weaker assumptions!

Marton's conjecture

SUMSET AND INVERSE SUMSET THEORY FOR SHANNON ENTROPY

TERENCE TAO

ABSTRACT. Let $G = (G, +)$ be an additive group. The sumset theory of Plünnecke and Ruzsa gives several relations between the size of sumsets $A + B$ of finite sets A, B, and related objects such as iterated sumsets kA and difference sets $A - B$, while the inverse sumset theory of Freiman, Ruzsa, and others characterises those finite sets A for which $A + A$ is small. In this paper we establish analogous results in which the finite set $A \subset G$ is replaced by a discrete random variable X taking values in G, and the cardinality $|A|$ is replaced by the Shannon entropy $\mathbf{H}(X)$. In particular, we classify the random variable X which have small doubling in the sense that $\mathbf{H}(X_1 + X_2) = \mathbf{H}(X) + O(1)$ when X_1, X_2 are independent copies of X, by showing that they factorise as $X = U + Z$ where U is uniformly distributed on a coset progression of bounded rank, and $\mathbf{H}(Z) = O(1)$.

When G is torsion-free, we also establish the sharp lower bound $\mathbf{H}(X+X) \geq \mathbf{H}(X) + \frac{1}{2}\log 2 - o(1)$, where $o(1)$ goes to zero as $\mathbf{H}(X) \to \infty$.

Marton's conjecture

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When G is torsion-free, we also establish the sharp lower bound $\mathbf{H}(X + X) \geq \mathbf{H}(X) + \frac{1}{2}\log 2 - o(1)$, where $o(1)$ goes to zero as $\mathbf{H}(X) \to \infty$.

SUMSETS AND ENTROPY REVISITED

BEN GREEN, FREDDIE MANNERS, AND TERENCE TAO

ABSTRACT. The entropic doubling $\sigma_{\mathrm{ent}}[X]$ of a random variable X taking values in an abelian group G is a variant of the notion of the doubling constant $\sigma[A]$ of a finite subset A of G, but it enjoys somewhat better properties; for instance, it contracts upon applying a homomorphism.

In this paper we develop further the theory of entropic doubling and give various applications, including:

(1) A new proof of a result of Pálvölgyi and Zhelezov on the "skew dimension" of subsets of $\mathbf{Z}^D$ with small doubling;

(2) A new proof, and an improvement, of a result of the second author on the dimension of subsets of $\mathbf{Z}^D$ with small doubling;

(3) A proof that the Polynomial Freiman–Ruzsa conjecture over $\mathbf{F}_2$ implies the (weak) Polynomial Freiman–Ruzsa conjecture over $\mathbf{Z}$.

Marton's conjecture

SUMSET AND INVERSE SUMSET THEORY FOR SHANNON ENTROPY

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ON A CONJECTURE OF MARTON

W. T. GOWERS, BEN GREEN, FREDDIE MANNERS, AND TERENCE TAO

ABSTRACT. We prove a conjecture of K. Marton, widely known as the polynomial Freiman–Ruzsa conjecture, in characteristic 2. The argument extends to odd characteristic, with details to follow in a subsequent paper.

Marton's Conjecture

Entropic Marton's Conjecture in $\mathbb{F}_2^n$:

Theorem (Gowers, Green, Manners, Tao, 2023)

If X is a RV on $\mathbb{F}_2^n$, then there is a subgroup H such that

$$H(X + U_H) - \frac{1}{2}H(U_H) - \frac{1}{2}H(X) \leq \frac{11}{2}(H(X_1 + X_2) - H(X))$$

Stability in continuous EPI?

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- $$d(X, Z) \leq C\delta_{\mathrm{EPI}}? \quad \text{Quantitative stability}$$

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- Stability fails for Wasserstein–2 and relative entropy [Courtade, Fathi, Pananjady 2018] $f(x) = \epsilon\sqrt{\frac{\epsilon}{\pi}}e^{-\frac{\epsilon x^2}{2}} + (1-\epsilon)\sqrt{\frac{1-\epsilon}{\pi}}e^{-\frac{(1-\epsilon)x^2}{2}}$

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Affirmative Resolution of Bourgain's Slicing Problem using Guan's Bound

Boaz Klartag and Joseph Lehec

Abstract

We provide the final step in the resolution of Bourgain's slicing problem in the affirmative. Thus we establish the following theorem: for any convex body $K \subseteq \mathbb{R}^n$ of volume one, there exists a hyperplane $H \subseteq \mathbb{R}^n$ such that

$$Vol_{n-1}(K \cap H) > c,$$

where $c > 0$ is a universal constant. Our proof combines Milman's theory of M-ellipsoids, stochastic localization with a recent bound by Guan, and stability estimates for the Shannon-Stam inequality by Eldan and Mikulincer.

“Stability” in the discrete EPI

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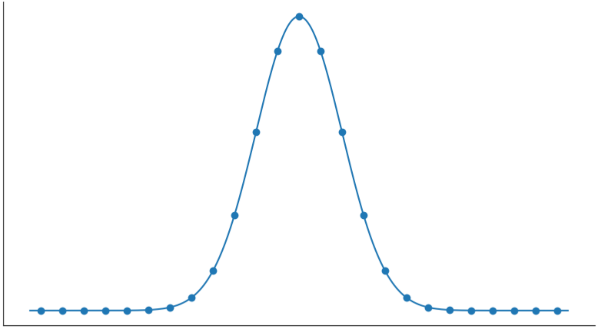
Theorem (G., Kontoyiannis 2025)

Suppose X has a log-concave distribution on the integers. Let $Z^{(\mathbb{Z})}$ a Gaussian with the same mean and variance as X, discretised on $\mathbb{Z}$. Then there are absolute constants C_1, C_2 such that

$$D(X||Z^{(\mathbb{Z})}) \leq C_1\left(H(X_1+X_2) - H(X_1) - \frac{1}{2}\log 2\right) + C_2\frac{\log\left(\mathrm{Var}(X)\right)}{\mathrm{Var}(X)}$$

Discretized Gaussian



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Theorem (G., Kontoyiannis 2025)

Assume X_1, X_2 IID and

$$\mathbb{E}|X_1|^k < \infty \text{ for some } k > 1.$$

Then

$$\text{for each } \epsilon > 0 \text{ there is a } \delta > 0 \text{ s.t. } \delta_{\mathrm{EPI}} < \delta$$

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d_{L} is the Lévy metric that metrizes convergence in distribution

Proof sketch

$$\delta_{\mathrm{EPI}}(X_1, X_2) := h(X_1 + X_2) - h(X_1) - \frac{1}{2}\log 2$$

Assume $\delta_{\mathrm{EPI}}(X_{1,n}, X_{2,n}) \to 0$ but $d_L(X_{1,n}, G) > c > 0$ for all Gaussians G

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$$\tilde{X}_{1,n} = \sqrt{1-t}X_{1,n} + \sqrt{t}Z_1, \quad \text{with } Z_1 \text{ standard Gaussian}$$

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  But then $\delta_{\mathrm{EPI}}(\tilde{X}_{1,n_\infty}, \tilde{X}_{2,n_\infty}) = 0$ without $\tilde{X}_{1,n_\infty}$ being Gaussian

Contradiction

Thank you!

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